Lecture 12

Wednesday, November 13, 2019 5:31 AM

## Path/line integrals.

(New) Defs. • A path/curve $\gamma : [a,b] \to G \subseteq \mathbb{C}$ is <u>smooth</u> if $\gamma$ is $C^1$.

(From now on, no need to assume $\gamma' \neq 0$; but $\gamma$ may not "look" smooth.)

• $\gamma$ is <u>piecewise smooth</u> if $\exists$ partition $a = a_0 < a_1 < \dots < a_n = b$ s.t. $\gamma : [a_{k-1}, a_k] \to G$ is smooth for $k = 1, \dots, n$.

• The <u>trace</u> of $\gamma$, denoted $\{\gamma\}$, is the image $\{\gamma\} = \gamma([a,b]) \subseteq G$.

• If $f$ is continuous on $\{\gamma\}$, $\gamma$ smooth, then

$$\int_\gamma f\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \quad \text{(Standard Riemann integral).}$$

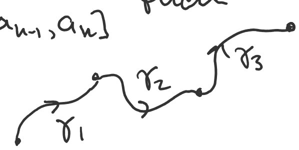
<u>Note</u>: In this course we will only deal with piecewise smooth curves. Rectifiable curves only add technical details and no useful additional information for the purposes of this course.

<u>Standard calculus applies</u>: $f, g$ cont., $\gamma : [a,b] \to G \subseteq \mathbb{C}$ smooth.

• $\int_\gamma (f+g)\,dz = \int_\gamma f\,dz + \int_\gamma g\,dz$, $\int_\gamma cf\,dz = c\int_\gamma f\,dz$ if $c \in \mathbb{C}$.

• If $a = a_0 < a_1 < \dots < a_n = b$ and $\gamma_k := \gamma|_{[a_{k-1}, a_k]}$ then

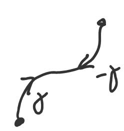
$$\int_\gamma f\,dz = \sum_{k=1}^n \int_{\gamma_k} f\,dz.$$




<u>Def.</u> • If $\gamma$ piecewise smooth, $\gamma_k = \gamma|_{[a_{k-1}, a_k]}$ smooth, then set

$$\int_\gamma f\,dz = \sum_{k=1}^n \int_{\gamma_k} f\,dz$$

• Let $(-\gamma)$ be the curve $-\gamma : [-b, -a] \to G \subseteq \mathbb{C}$, $(-\gamma)(t) = \gamma(-t)$.



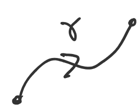

<u>Note</u>:
$$\int_{-\gamma} f\,dz = -\int_{-b}^{-a} f(\gamma(-t))\,\gamma'(-t)\,dt = \{s = -t\} = -\int_a^b f(\gamma(s))\gamma'(s)\,ds$$
$$= -\int_\gamma f\,dz.$$

$$= -\int_{\gamma} f\,dz.$$

**Prop 1.** (Independence of parametrization). Let $\gamma:[a,b]\to G\subseteq\mathbb{C}$ be piecewise smooth curve, $\varphi:[c,d]\to[a,b]$ a $C^1$ nondecreasing function (reparametrization), and let $\tilde\gamma=\gamma\circ\varphi:[c,d]\to G$.

Then, $$\int_{\tilde\gamma} f\,dz=\int_{\gamma} f\,dz.$$

**Remark:** We will say two curve $\gamma,\tilde\gamma$ are equivalent if $\exists$ such $\varphi$. Prop 1 says equivalent curves produce equal integrals. Thus, to determine $\int_\gamma f\,dz$, we need only know $\{\gamma\}$ and direction traversed





**Pf of Prop 1.** Suffices to consider smooth $\gamma$.

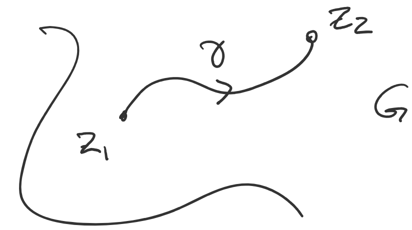
$$\int_\gamma f\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)\,dt=\left\{\begin{array}{l|l} t=\varphi(s) & d\to b\\ dt=\varphi'(s) & c\to a\end{array}\right\}=\int_c^d f(\gamma(\varphi(s)))\,\gamma'(\varphi(s))\,\varphi'(s)\,ds$$

$$=\int_c^d f(\tilde\gamma(s))\,\tilde\gamma'(s)\,ds=\int_{\tilde\gamma} f\,dz. \qquad \blacksquare$$

**Prop 2.** (FTC in $\mathbb{C}$). Assume $\exists$ function $F$ in $G$ $\subseteq\mathbb{C}$ s.t. $F'=f\in\mathcal{C}(G)$. $\Rightarrow F$ is analytic

Then, if $\gamma$ is p-w smooth curve in $G$, from $z_1=\gamma(a)$, $z_2=\gamma(b)$,

$$\int_\gamma f\,dz=F(z_2)-F(z_1)$$





**Pf.** Standard FTC:

$$\int_\gamma f\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)\,dt=\left\{\frac{d}{dt}F(\gamma(t))=F'(\gamma(t))\,\gamma'(t)=f(t)\,\gamma'(t)\right\}$$

$$\int_\gamma f\,dz = \int_a \ldots$$

$$= \int_a^b \frac{d}{dt}(F\circ\gamma)\,dt = \{FTC\} = (F\circ\gamma)(b) - (F\circ\gamma)(a) = F(z_2) - F(z_1).$$

$\square$

<u>Thm</u> (Leibniz rule) Let $\varphi : [a,b]\times[c,d]$ be cont. and $\frac{\partial\varphi}{\partial t}$ cont. Consider

$$g(t) = \int_a^b \varphi(s,t)\,ds.$$

Then, $g$ is $\mathcal{C}^1$ and $g'(t) = \int_a^b \frac{\partial\varphi}{\partial t}(s,t)\,ds$

<u>Pf:</u> $$\frac{g(t+h)-g(t)}{h} - \int_a^b \varphi_t(s,t)\,ds = \int_a^b \left[\frac{\varphi(s,t+h)-\varphi(s,t)}{h} - \varphi_t(s,t)\right] ds \quad (1)$$

Since $t \to \varphi(s,t)$ is $\mathcal{C}^1$ $\overset{MVT}{\Rightarrow}$ $\exists$ $h(s)$ w/ $|h(s)| < |h|$ s.t.

$$\frac{\varphi(s,t+h)-\varphi(s,t)}{h} = \varphi_t(s,t+h(s)). \quad (2)$$

Moreover, since $\varphi_t$ is cont. on compact $[a,b]\times[c,d]$ it is <u>unif. cont.</u> $\Rightarrow$ $\forall \varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $|(s'-s, t'-t)| < \delta \Rightarrow$ $|\varphi_t(s',t') - \varphi_t(s,t)| < \frac{1}{b-a}\varepsilon$. But if $|h| < \delta$ then $|(s,t+h(s)) - (s,t)| = |(0,h(s))| < |h| < \delta \Rightarrow$ by (1), (2)

$$\left|\frac{g(t+h)-g(t)}{h} - \int_a^b \varphi_t(s,t)\,ds\right| \leq \int_a^b |\varphi_t(s,t+h(s)) - \varphi_t(s,t)|\,ds$$

$$< \varepsilon.$$

I.e. $$\lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \int_a^b \varphi_t(s,t)\,ds. = g'(t)$$

To see $g'(t)$ cont. use a similar argument using again that $\varphi_t$ is cont. in $(s,t)$.

To see $g(t)$ cont., a simila... is cont. in $(s,t)$.

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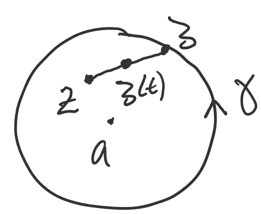
## Power series representations of analytic fcns

**Prop 3** (Baby Cauchy Integral Formula). Let $f$ be analytic in $G \subseteq \mathbb{C}$ and assume $\overline{B(a,r)} \subset G$. Let $\gamma : [0,2\pi] \to G$ be circle $|z-a| = r$ traversed in positive direction ($\gamma(t) = a + re^{it}$). Then, for $z \in B(a,r)$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

**Pf.** Fix $z \in B(a,r)$ and define $g(t) = \frac{1}{2\pi i} \int_\gamma \frac{f(\overbrace{z+t(\zeta-z)}^{\zeta(t)})}{\zeta - z}\,d\zeta - f(z)$, $t \in [0,1]$.




$\Rightarrow$

$$g(t) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + t(a+re^{is} - z))}{a + re^{is} - z}\, ire^{is}\,ds.$$

By Leibniz rule, $g \in \mathcal{C}^1$ and $g'(t) = \frac{1}{2\pi i} \int_0^{2\pi} \underbrace{f'(z+t(a+re^{is}-z))\, ire^{is}}_{\frac{1}{t}\frac{d}{ds}\left[f(z+t(a+re^{is}-z))\right]}\,ds$

$$= \frac{1}{2\pi i t} \int_0^{2\pi} \frac{d}{ds}\Big(f\big(z+t(a+re^{is}-z)\big)\Big)\,ds = \frac{1}{2\pi i t}\Big[f(z+t(a+r-z)) - f(z+t(a+r-z))\Big] = 0$$

Thus, $g(1) = g(0) \Rightarrow$

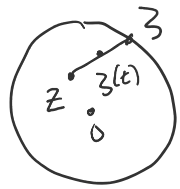
$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z) = f(z)\left(\frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} - 1\right). \quad (3)$$

Prop 3 now follows from the following lemma.

<u>Lemma 1</u>. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be circle $|z-a| = r$ traversed in positive direction. Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} = 1 \quad \text{if} \quad |z-a| < r.$$

<u>Pf</u>. By setting $w = \frac{z-a}{r}$, we can reduce $\overset{\text{WLOG}}{\text{to}}$ the case $a = 0$, $\gamma$ being unit circle $|z| = 1$. We utilize Leibniz as above. Set $g(t) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - \underbrace{tz}_{\zeta(t)}}$, $t \in [0,1]$.




Then, by Leibniz, $g'(t) = \frac{1}{2\pi i}\int_\gamma \frac{z\,d\zeta}{(\zeta - tz)^2} = \frac{1}{2\pi i}\int_\gamma \frac{d}{d\zeta}\left(\frac{1}{\zeta - tz}\cdot\left(-\frac{1}{z}\right)\right)d\zeta$

$= 0$ since $\gamma$ is closed $(\gamma(0) = \gamma(2\pi))$.

Thus $g(1) = g(0) \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{i\cancel{e^{is}}}{\cancel{e^{is}}}\,ds = 1.$

□

This proves Prop 3 by (3).